LINEARLY-CONTROLLED ASYMPTOTIC DIMENSION OF THE FUNDAMENTAL GROUP OF A GRAPH-MANIFOLD

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Abstract. We prove the estimate $\ell\text{-asdim}\,\pi_1(M) \leq 7$ for the linearly controlled asymptotic dimension of the fundamental group of any 3-dimensional graph-manifold M. As applications we obtain that the universal cover $\widetilde{M}$ of M is an absolute Lipschitz retract and it admits a quasisymmetric embedding into the product of 8 metric trees.

1. Introduction

A motivation of our work is a result of Bell and Dranishnikov [3, Theorem 1′] that the asymptotic dimension of a graph-group, whose vertex groups have finite asymptotic dimensions, is also finite. The fundamental groups of graph-manifolds are graph-groups. However, it is unclear whethere there exists an analogue of the Bell-Dranishnikov result for the linearly-controlled asymptotic dimension. We present here a construction of suitable coverings for the universal cover of a graph-manifold that allows us to prove the following:

Theorem 1. *For the fundamental group of a graph-manifold M taken with any word metric and its linearly-controlled asymptotic dimension, we have* $\ell\text{-asdim}\,\pi_1(M) \leq 7$.

Corollary 1. *Let $\widetilde{M}$ be the universal cover of a graph-manifold M. Then $3 \leq \dim_{\mathrm{AN}} \widetilde{M} \leq 7$, where $\dim_{\mathrm{AN}}$ is the Assouad-Nagata dimension, M is taken with any Riemannian metric and $\widetilde{M}$ with the metric lifted from M.*

As applications of Corollary 1 and [9, Theorem 1.3, Theorem 1.5], we obtain.

Corollary 2. *Let $\widetilde{M}$ be the universal cover of a graph-manifold M taken with a Riemannian distance d lifted from M. Then for all sufficiently small $p \in (0,1)$, there exists a bi-Lipschitz embedding of $(\widetilde{M}, d^p)$ into the product of 8 metric trees, in particular $(\widetilde{M}, d)$ admits a quasisymmetric embedding into the product of 8 metric trees.*

Corollary 3. *The universal cover $\widetilde{M}$ of a graph-manifold is an absolute Lipschitz retract, that is, given a metric space X, there is $C > 0$ such that for every subset $A \subset X$ and every λ-Lipschitz map $f\colon A \to \widetilde{M}$, $\lambda > 0$, there exists a $C\lambda$-Lipschitz extension $\bar{f}\colon X \to \widetilde{M}$ of f.*

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2. Preliminaries

In this section we recall some basic definitions and notations. Let X be a metric space. We denote by $|xy|$ the distance between $x, y \in X$, and $d(U,V) := \inf\{|uv| \mid u \in U, v \in V\}$ is the distance between $U, V \subset X$. We write $B_r(x) = \{x' \in X \mid |xx'| \le r\}$ for the ball with the center x and the radius r. A map $f\colon X \to Y$ is said to be *quasi-isometric* if there exist $\lambda \ge 1, C \ge 0$ such that

$$\frac{1}{\lambda}|xy| - C \le |f(x)f(y)| \le \lambda|xy| + C$$

for each $x, y \in X$. Metric spaces X and Y are called *quasi-isomeric* if there is a quasi-isometric map $f\colon X \to Y$ such that $f(X)$ is a net in Y. In this case f is called a *quasi-isometry.*

We say that a family $\mathcal{U}$ of subsets of X is a *covering* if for each point $x \in X$ there is a subset $U \in \mathcal{U}$ such that $x \in U$. A family $\mathcal{U}$ of sets is *disjoint* if each two sets $U, V \in \mathcal{U}$ are disjoint. The union $\mathcal{U} = \cup\{\mathcal{U}^\alpha \mid \alpha \in \mathcal{A}\}$ of disjoint families U^α is said to be *n-colored*, where $n = |\mathcal{A}|$ is the cardinality of $\mathcal{A}$.

Also recall that a family $\mathcal{U}$ is *D-bounded*, if the diameter of every $U \in \mathcal{U}$ does not exceed D, $\operatorname{diam} U \le D$. An n-colored family of sets $\mathcal{U}$ is *r-disjoint*, if for every color $\alpha \in \mathcal{A}$ and each two sets $U, V \in \mathcal{U}^\alpha$ we have $d(U,V) \ge r$.

The linearly-controlled asymptotic dimension is a version of the Gromov's asymptotic dimension, asdim.

Definition 1. (Roe [10]) The *linearly-controlled asymptotic dimension* of a metric space X, $\ell\text{-asdim}\, X$, is the least integer number n such that for each sufficiently large real R there exists an $(n+1)$-colored, R-disjoint, CR-bounded covering of the space X, where the number $C > 0$ is independent of R.

A *tripod* in a geodesic metric space X is a union of three geodesic segments $xt \cup yt \cup zt$ which have only one common point t. We say that t is the *center* of the tripod. A geodesic metric space X is called a *metric tree* if each triangle in it is a tripod (possibly degenerate).

Let T be a metric tree. Recall the well-known construction of 2-colored, r-disjoint, $3r$-bounded covering of T (see [10]). Fix some vertex $v \in T$. Suppose $k \ge 1$ is an integer and consider the annuli

$$A_k := \{x \in T \mid kr \le |vx| \le (k+1)r\}$$

of the width r and with the center v. For $x, y \in A_k$ we say that $x \sim_k y$ if $|zv| \ge (k - \frac{1}{2})r$, where z is the center of the tripod with vertices x, y, v. One can easily check (see [10]) that $\sim_k$ is an equivalence relation on A_k. We will make use of the following 2-colored covering: $\mathcal{U} := \mathcal{U}^1 \cup \mathcal{U}^2$ where $\mathcal{U}^1$ consists of the ball $B_r(v)$ and the equivalence classes for $\sim_k$ corresponding to all even k. The family $\mathcal{U}^2$ consists of the equivalence classes for $\sim_k$ corresponding to all odd k. It is shown in [10] that this covering is r-disjoint and $3r$-bounded. We call it the *standard (r, v)-covering* of T.

We also use a modified covering where the ball $B_r(v)$ is replaced by the ball $B_d(v)$ with $0 < d \le r$. The annuli A_k are replaced by the annuli

$$B_k := \{x \in T \mid (k-1)r + d \le |vx| \le kr + d\}$$

of the width r and with the center v. Now for each integer $k > 1$ as well as for $k = 1$ and $d \geq r/2$, we introduce on B_k the equivalence relation $x \sim_k y$ iff $|zv| \geq (k - \frac{3}{2})r + d$. If $k = 1$ and $d < r/2$, then we say that all points in B_1 are $\sim_1$-equivalent. As before one can easily check that the modified covering is r-disjoint and $3r$-bounded. We call this covering the *standard* (r, d, v)*-covering* of T.

Let G be a finitely generated group and $S \subset G$ a finite symmetric generating set for G ($S^{-1} = S$). Recall that a *word metric* on the group G (with respect to S) is the metric defined by the norm $\|\cdot\|_S$ where for each $g \in G$ its norm $\|g\|_S$ is the smallest number of elements of S whose product is g. In this paper we will consider only finitely generated groups with a word metric.

Definition 2. By a *graph-manifold* we mean a closed three dimensional manifold which admits a finite cover by a connected orientable manifold that is glued from blocks along boundary tori in a regular way. By a *block* we mean a trivial S^1-fibration over a surface with negative Euler characteristic and nonempty boundary. A gluing of blocks M_1 and M_2 (possibly $M_1 = M_2$) along boundary tori $T_1 \subset \partial M_1$ and $T_2 \subset \partial M_2$ is said to be *regular*, if S^1-fibers, coming up from T_1 and T_2, are not homotopic on the gluing torus.

Fix a Riemannian metric on a graph-manifold M. The following fact is well known [5].

Lemma 1. *Let Y be a compact metric space with length metric. Let X be a universal cover of Y with the metric lifted from Y. Then X is quasi-isometric to the group $\pi_1(Y)$ with any word metric.*

3. Graph-manifold model

3.1. **Description of the model.** Consider the hyperbolic plane $\mathbb{H}^2_\kappa$ having a curvature $-\kappa$ ($\kappa > 0$) such that the side of a rectangular equilateral hexagon θ in the plane $\mathbb{H}^2_\kappa$ has the length 1. Let ρ be the distance between the middle points of sides, which have a common adjacent side, δ the diameter of θ. It is clear that $\rho \leq 2$, $\delta \leq 3$.

We mark each second side of θ (so we have marked three sides) and consider a set H_0 defined as follows. Take the subgroup G_θ of the isometry group of $\mathbb{H}^2_\kappa$ generated by reflections in (three) marked sides of θ and let H_0 be the orbit of θ with respect to G_θ. Then H_0 is a convex subset in $\mathbb{H}^2_\kappa$ divided into hexagons that are isometric to θ. Furthermore, the boundary of H_0 has infinitely many connected components each of which is a geodesic line in $\mathbb{H}^2_\kappa$.

The graph T_0 dual to the decomposition of H_0 into hexagons is the standard binary tree whose vertices all have degree three. Any metric space isometric to H_0 will be called a *θ-tree*. Given a vertex p of T_0, we denote by θ_p the respective hexagon in H_0.

Every boundary line ℓ of H_0 is the union of segments of length one each of which is a side of a hexagon in H_0. Thus ℓ determines a sequence of vertices of T_0 consecutively connected by edges which form a line $\bar{\ell}$ in T_0.

Consider the line $\mathbb{R}$ and its partition into segments $\mathbb{R} = \bigcup\{[i, i+1] \mid i \in \mathbb{Z}\}$. Then the graph S_0 dual to this partition is an infinite tree with all the

vertices having degree 2, in particular, S_0 is homeomorphic to $\mathbb{R}$. Given a vertex q of S_0, we denote by $I_q \subset \mathrm{H}_0$ the respective segment.

Remark 1. In what follows we will consider T_0 and S_0 as metric spaces with metrics such that the length of each edge is equal to 1. Then these metric spaces are metric trees. We will denote the set of vertices in T_0 by $\mathrm{V}(T_0)$, and the set of vertices in S_0 by $\mathrm{V}(S_0)$.

Remark 2. Given distinct vertices $p, q \in \mathrm{V}(T_0)$ in T_0 and $x \in \theta_p, y \in \theta_q$, we have

$$|pq| - 1 \le |xy| \le (|pq| - 1)\rho + 2\delta. \tag{$\star$}$$

Similarly, given distinct $p, q \in \mathrm{V}(S_0)$ and $x \in I_p, y \in I_q$, we have

$$|pq| - 1 \le |xy| \le |pq| + 1. \tag{$\star\star$}$$

Remark 3. Note that for each connected subset $U \subset T_0$ the set $U' := \cup\{\theta_p \mid p \in U \cap \mathrm{V}(T_0)\}$ is convex. Similarly, for each connected subset $V \subset S_0$ the set $V' := \cup\{I_q \mid q \in V \cap \mathrm{V}(S_0)\}$ is convex.

Now, we describe the following metric space X.

Construction. Let T be an infinite tree all whose vertices have countably infinite degree. Denote by $\mathrm{V}(T)$ the set of all the vertices of T and by $\mathrm{E}(T)$ the set of all the edges of T. For each $v \in \mathrm{V}(T)$ consider the metric product $X_v := \mathrm{H}_0 \times \mathbb{R}$ and fix a bijection b_v between the set E_v of all the edges adjacent to v and all the boundary lines in the factor H_0. Note that the partition of H_0 into hexagons and the partition of $\mathbb{R}$ into segments give rise to a partition of X_v into sets isometric to $\theta \times [0, 1]$ which we call *bricks*. Furthermore, each boundary plane $\sigma \subset X_v$ is obtained as the product of a boundary line ℓ in the θ-tree factor and the factor $\mathbb{R}$, and the partition of X_v into bricks gives rise to a partition of σ into squares with side length 1, which we call *the grid* on the plane σ. Next, consider for each edge uv in $\mathrm{E}(T)$ the boundary planes $\sigma_u = b_u(uv) \times \mathbb{R}_u \subset X_u$ and $\sigma_v = b_v(uv) \times \mathbb{R}_v \subset X_v$, respectively, and glue them by an isometry that glues the grids on σ_u and σ_v while flips the factors $\mathbb{R}_u$, $\mathbb{R}_v$ to the factors $b_v(uv)$, $b_u(uv)$ respectively. We call the obtained space the *model* and denote it by X.

Recall that a complete geodesic metric space Y is called a CAT(0)-*space*, if the following is satisfied: for each triple of points $x, y, z \in X$ and a point $t \in yz$ consider points $\bar{x}, \bar{y}, \bar{z}$ on the plane $\mathbb{R}^2$ such that $|xy| = |\bar{x}\bar{y}|$, $|yz| = |\bar{y}\bar{z}|$, $|xz| = |\bar{x}\bar{z}|$ and the point $\bar{t} \in [\bar{y}\bar{z}]$, such that $|yt| = |\bar{y}\bar{t}|$, then we have $|xt| \le |\bar{x}\bar{t}|$. A CAT(0)-space is also called an *Hadamard space.* The spaces H_0 and $\mathbb{R}$ are CAT(0)-spaces, so X_v, $v \in \mathrm{V}(T)$ and X are CAT(0)-spaces too (see for example [5]).

3.2. **The model X as the universal cover of a closed graph-manifold.** Here we construct a closed graph-manifold Q (with C^1-smooth Riemannian nonpositively curved metric) whose universal cover $\widetilde{Q}$ is isometric to X.

Let ω be the union of adjacent hexagons $\theta_1, \theta_2 \subset \mathrm{H}_0$. Denote by $\tau\colon \theta_1 \to \theta_2$ the reflection with respect to the common side $\theta_1 \cap \theta_2$ and consider the surface P_0, obtained from ω by gluing the marked sides matched by τ. Then P_0 is a surface of constant curvature $-\kappa$ with geodesic boundary. The boundary ∂P_0 consist of 3 components γ_1, γ_2 and γ_3 of length 2. The

hexagon vertices subdivide every γ_i, $i = 1, 2, 3$, into a pair of segments of length 1.

Let us mark the images of hexagons' vertices (so, there will be 6 marked points). Note that the universal cover of P_0 is isometric to H_0. Consider the circle S^1_2 of length 2 and mark two diametrally opposite points s_1 and s_2 on it. For each boundary component γ_i ($i \in \{1, 2, 3\}$) consider an isometry $\Phi_i\colon \gamma_i \to S^1_2$ taking the marked points into the marked points.

Let $P := P_0 \times S^1_2$. Consider an isometric copy P' of P. All the elements of P' will be denoted by the same letters but with prime. Let Q be the manifold obtained by gluing P and P' along the boundary tori $\gamma_i \times S^1_2 \subset \partial P$ and $S^1_2 \times \gamma'_i \subset \partial P'$ according to isometries

$$\Phi_i \times \Phi'^{-1}_i\colon\ \gamma_i \times S^1_2 \longrightarrow S^1_2 \times \gamma'_i,\ i \in \{1, 2, 3\}.$$

Note that Q is a graph-manifold the universal cover $\widetilde{Q}$ of which consist of blocks isometric to $\mathrm{H}_0 \times \mathbb{R}$. Any two of these blocks either are disjoint or intersect over a plane covering a torus $\gamma_i \times S^1_2$, $i \in \{1, 2, 3\}$. This plane is divided by the lines covering the circles $\{x_i\} \times S^1_2$ and $\gamma_i \times \{s_i\}$ (where $x_i \in \gamma_i$ and $s_i \in S^1_2$ are marked points) into squares with side 1. It is clear that $\widetilde{Q}$ isometric to X. We recall the following result [2, Theorem 2.1].

Theorem 2. *Any two graph-manifolds have bi-Lipschitz homeomorphic universal covers, in particular, their fundamental groups are quasi-isometric.*

It follows from this theorem that for any graph-manifold M its fundamental group $\pi_1(M)$ is quasi-isometric to the fundamental group $\pi_1(Q)$ of the manifold Q which is by Lemma 1 quasi-isometric to the model X. Therefore, since ℓ-asdim is a quasi-isometry invariant (see e.g. [9]), we have $\ell\text{-asdim}\,\pi_1(M) = \ell\text{-asdim}\,X$. So, to prove Theorem 2 it suffices to show that $\ell\text{-asdim}\,X \leq 7$. To this end, we construct for each $R \in \mathbb{N}$, $R > 10$ an R-disjoint, CR-bounded 8-colored covering of the model X, where $C = 88$.

4. Coverings of the model

We briefly describe the construction of the required covering of X. We fix $R \in \mathbb{N}$, $R > 10$. First, we construct for every block X_u, $u \in \mathrm{V}(T)$, a 4-colored, $2R$-disjoint, $15R$-bounded covering $\mathcal{W}_u$ which is the product of standard 2-colored coverings of the factors in the decomposition $X_u = \mathrm{H}_0 \times \mathbb{R}$. For different neighboring blocks these covering are compatible (see Lemma 2) and form together 4-colored covering $\widetilde{\mathcal{W}}$ of X. The covering $\widetilde{\mathcal{W}}$ is R-disjoint (see Proposition 1), and this is the main result of the current section. However, all the members of $\widetilde{\mathcal{W}}$ have infinite diameter. Fortunately, every member W of $\widetilde{\mathcal{W}}$ has a tree-like structure, and we construct in sect. 5.2 a 2-colored covering of W similar to the standard 2-colored covering of a tree. Typical W is disconnected having every connected component convex. For technical reasons, it is convenient first to extend every $W \in \widetilde{\mathcal{W}}$ to a convex $\overline{W} \supset W$ and then to cover $\overline{W}$ by a 2-colored $\mathcal{X}_W$. The extension procedure is described in sect. 5.1. The convexity plays an important role in proving the properties of $\mathcal{X}_W$. After that, we forget about the extension $\overline{W}$ and restrict $\mathcal{X}_W$ to W. In that way, we produce the required 8-colored covering $\mathcal{X}$ of X.

4.1. **Construction of a covering of a block.** Put $N := 2R+1$. Consider an arbitrary block $X_v = \mathrm{H}_0 \times \mathbb{R} \subset X$. We choose a brick $\theta_p \times I_q \subset X_v$, where $p \in \mathrm{V}(T_0)$ and $q \in \mathrm{V}(S_0)$. Consider the standard (N,p)-covering $\mathcal{U} = \mathcal{U}^1 \cup \mathcal{U}^2$ of the tree T_0 (see section 2). We build a 2-colored covering $\widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}^1 \cup \widetilde{\mathcal{U}}^2$ of the θ-tree as follows: for each set $U \in \mathcal{U}^i$, $i \in \{1,2\}$ we define the set

$$\widetilde{U} := \bigcup \{\theta_{p'} \mid p' \in U \cap \mathrm{V}(T_0)\}$$

and consider the families

$$\widetilde{\mathcal{U}}^i := \{\widetilde{U} \mid U \in \mathcal{U}^i\},\ i \in \{1,2\}.$$

It is clear that the family $\widetilde{\mathcal{U}}$ is a 2-colored covering of the θ-tree. Recall that ρ is the distance between the middle points of two sides of the hexagon θ, which have a common adjacent side, and δ is its diameter.

Since the covering $\mathcal{U}$ is N-disjoint and $3N$-bounded, the inequality $(\star)$ implies that the covering $\widetilde{\mathcal{U}}$ is $2R$-disjoint and $((6R+2)\rho + 2\delta)$-bounded. Since $\rho \leq 2, \delta \leq 3$ and $R > 10$ we have $(6R+2)\rho + 2\delta \leq 13R$ so $\widetilde{\mathcal{U}}$ is $2R$-disjoint and $13R$-bounded.

Fix a natural $d \leq N$. Consider the standard (N,d,q)-covering $\mathcal{V} = \mathcal{V}^1 \cup \mathcal{V}^2$ of the tree S_0 (see section 2). We build a 2-colored covering $\widetilde{\mathcal{V}} = \widetilde{\mathcal{V}}^1 \cup \widetilde{\mathcal{V}}^2$ of the line $\mathbb{R}$ as follows: for each set $V \in \mathcal{V}^i$, $i \in \{1,2\}$ consider the set

$$\widetilde{V} := \bigcup \{I_{p'} \mid p' \in V \cap \mathrm{V}(S_0)\}$$

and the families

$$\widetilde{\mathcal{V}}^i := \{\widetilde{V} \mid V \in \mathcal{V}^i\},\ i \in \{1,2\}.$$

It is clear that the family $\widetilde{\mathcal{V}}$ is a 2-colored covering of $\mathbb{R}$. Note, that since the family $\mathcal{V}$ is N-disjoint and $3N$-bounded the inequality $(\star\star)$ implies that the covering $\widetilde{\mathcal{V}}$ is $2R$-disjoint and $(6R+2)$-bounded. Since $R > 10$ we have $6R + 2 \leq 7R$, so the covering $\widetilde{\mathcal{V}}$ is $2R$-disjoint and $7R$-bounded.

Consider the 4-colored covering

$$\mathcal{W} := \bigcup \{\mathcal{W}^{(i,j)} \mid (i,j) \in \{1,2\} \times \{1,2\}\}$$

of the block X_v where

$$\mathcal{W}^{(i,j)} := \{U \times V \mid U \in \widetilde{\mathcal{U}}^i, V \in \widetilde{\mathcal{V}}^j\}.$$

Since the coverings $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ are $2R$-disjoint the covering $\mathcal{W}$ is also $2R$-disjoint, and since the covering $\widetilde{\mathcal{U}}$ is $13R$-bounded and $\widetilde{\mathcal{V}}$ is $7R$-bounded, the covering $\mathcal{W}$ is $R\sqrt{7^2 + 13^2}$-bounded, thus it is $15R$-bounded. We will call this covering of the block X_v the (N,d)-*covering with the initial brick* $\theta_p \times I_q$.

Consider the block X_v with the fixed initial brick $\theta_{p_v} \times I_{q_v}$, $p_v \in \mathrm{V}(T_0), q_v \in \mathrm{V}(S_0)$, and its boundary plane $\sigma = \ell \times \mathbb{R}_v$, where ℓ is a boundary line of the θ-tree factor of X_v. Let $\bar{\ell}$ be the line in the tree T_0 that corresponds to ℓ (see section 3.1), $p' \in \bar{\ell}$ the nearest to p_v vertex. We denote $n := |p_v p'|$ and put $k \equiv n \pmod{N}$, $0 \leq k < N$.

Let X_u be the block glued to σ, $1 \leq d \leq N$ a natural number, $I_{q_u} \times \theta_{p_u}$ the brick in X_u having the common boundary square with the brick $\theta_{p'} \times I_{q_v}$. We represent the boundary plane σ as the product of the line $\mathbb{R}$ and an

appropriate boundary line ℓ' in the θ-tree factor of the block X_u, $\sigma = \mathbb{R}_u \times \ell'$. Consider the line $\bar{\ell}'$ in the tree T_0 which corresponds to the line ℓ'. Then the vertex p_u belongs to $\bar{\ell}'$. We choose a vertex p'_u of T_0 such that the vertex p_u is the nearest to p'_u point of the line $\bar{\ell}'$ and the distance $|p_u p'_u|$ is equal to $N - d$.

Lemma 2. *The (N, d)-covering $\mathcal{W}_v$ with the initial brick $\theta_{p_v} \times I_{q_v}$ of the block X_v and the $(N, N-k)$-covering $\mathcal{W}_u$ with the initial brick $I_{q_u} \times \theta_{p'_u}$ of the block X_u agree on the common boundary plane σ of the blocks X_u and X_v, i.e. for each pair of numbers $(i,j) \in \{1,2\} \times \{1,2\}$ any two sets $V \in \mathcal{W}_v^{(i,j)}$ and $U \in \mathcal{W}_u^{(j,i)}$ either are disjoint or their intersections with σ coincide.*

Proof. Indeed since the coverings $\mathcal{W}_v$ and $\mathcal{W}_u$ are products of the coverings of the corresponding factors, it is enough to check that the coverings of the factors agree. We have $\sigma = \ell \times \ell'$ and the $\mathbb{R}$-factor of the block X_u is glued to the line ℓ. Let $\bar{\ell}$ be the line corresponding to the line ℓ in the tree T_0. If we consider the line $\bar{\ell}$ as an isometric copy of the tree S_0 then the standard (N, p_v)-covering of the tree T_0 induces a covering $\mathcal{U}'$ on the line $\bar{\ell}$ which is a standard $(N, N-k, p')$-covering. Therefore since the side of the hexagon $\theta_{p'}$ lying on ℓ is glued to the segment I_{q_u} of the $\mathbb{R}$-factor of the block X_u, the standard $(N, N-k, q_u)$-covering of this $\mathbb{R}$-factor coincides with the covering $\mathcal{U}'$. The case of the other factors is similar. □

4.2. **Constructing compatible coverings of blocks.** Recall that T is the canonical simplicial tree with all the vertices having the infinite countable degree. We consider the metric on T with respect to which all the edges have length one. We fix an arbitrary vertex $v \in \mathrm{V}(T)$ and call it the *root* of the tree T. We define the *level* function $l : \mathrm{V}(T) \to \mathbb{Z}$ by $l(v') := |vv'|$. We build coverings of blocks of the type discussed earlier which agree with the coverings of the neighbor blocks on the common boundary planes (see sect. 4.1) by induction on the level m.

The base of induction : $m = 0$.

Let $\theta_p \times I_q$ be an arbitrary brick in X_v. Consider the (N, N)-covering with the initial brick $\theta_p \times I_q$ of the block X_v. We denote this covering by $\mathcal{W}_v$.

The induction step: $m \to m + 1$.

Suppose that we have already built the coverings $\mathcal{W}_{v'}$ which are compatible for all the blocks $X_{v'}$ such that $l(v') \leq m$. Consider an arbitrary vertex $u \in \mathrm{V}(T)$ such that $l(u) = m + 1$. The block X_u has exactly one boundary plane σ along which it is glued with a block X_w such that $l(w) = m$. X_w is already covered by $\mathcal{W}_w$ which is an (N, d_w)-covering with the initial brick $\theta_{p_w} \times I_{q_w}$. Consider the plane $\sigma = \ell \times \mathbb{R}$ as the product of a boundary line ℓ in the θ-tree and the factor $\mathbb{R}$. Let $\bar{\ell}$ be the line in T_0 corresponding to ℓ (see section 3.1). We take the vertex p' in $\bar{\ell}$ which is the nearest to p_w and denote $n_w := |p_w p'|$. Let $d_u \equiv n_w \pmod N$, $0 \leq d_u < N$. Let $I_{q_u} \times \theta_{p_u}$ be a brick in X_u having a common boundary square with the brick $\theta_{p'} \times I_{q_w}$. We represent the boundary plane σ as the product of the line $\mathbb{R}$ and a boundary line in the θ-tree factor of the block X_u, $\sigma = \mathbb{R} \times \ell'$. Consider the line $\bar{\ell}'$ in T_0 which corresponds to the line ℓ'. It is clear that the vertex p_u belongs to $\bar{\ell}'$. We choose a vertex p'_u in T_0 such that the vertex p_u is the nearest to p'_u

point on the line $\bar{\ell}'$ and the distance $|p_u p'_u|$ is equal to $N - d_w$. We denote by $\mathcal{W}_u$ the $(N, N-d_u)$-covering of the block X_u with the initial brick $I_{q_u} \times \theta_{p'_u}$. It follows from Lemma 2 that the coverings $\mathcal{W}_w$ and $\mathcal{W}_u$ of the blocks X_w and X_u respectively agree on the boundary plane σ. $\mathcal{W}_u$ is the covering of the type we need so the induction step is completed.

4.3. **Constructing a covering of the model.** Consider the following 4 families of subsets of the model X. For each pair of numbers $(i,j) \in \{1,2\} \times \{1,2\}$ we denote

$$\mathcal{W}^{(i,j)} := \{U \mid U \in \mathcal{W}_u^{(i,j)},\ l(u) \text{ is even}\} \cup \{U \mid U \in \mathcal{W}_u^{(j,i)},\ l(u) \text{ is odd}\}.$$

Moreover, for each pair of numbers $(i,j) \in \{1,2\} \times \{1,2\}$ we consider the following relation $\sim_{(i,j)}$ on the family $\mathcal{W}^{(i,j)}$: $U \sim_{(i,j)} V$ iff there are sets $U_0 = U, U_1, \dots, U_{n+1} = V$ in the family $\mathcal{W}^{(i,j)}$ such that $U_j \cap U_{j+1} \neq \emptyset$ for all $j \in \{0, \dots, n\}$. It is clear that $\sim_{(i,j)}$ is an equivalence relation.

Proposition 1. *Suppose $W, U \in \mathcal{W}^{(i,j)}$ and $d(W,U) < R$. Then $W \sim_{(i,j)} U$.*

We need two facts (see Lemma 3 and 4) for the proof of Proposition 1.

Lemma 3. *Given $U \in \mathcal{W}^{(i,j)}$, $U \subset X_u$, assume that a boundary plane $\sigma \subset X_u$ meets U, $U \cap \sigma \neq \emptyset$, and separates a point $x \in X$ and U. Then $d(x,U) = d(x, U \cap \sigma)$.*

Proof. Without loss of generality, we assume that $x \notin \sigma$. Then, since the hyperplane $\sigma \subset X$ is convex, the segment xy meets σ over a point for every $y \in U$, $xy \cap \sigma = t$.

Recall that we represent the block X_u as the product $X_u = \mathrm{H}_0 \times \mathbb{R}$ and also $U = \widetilde{U}_\theta \times \widetilde{U}_\mathbb{R}$, where $\widetilde{U}_\theta$, $\widetilde{U}_\mathbb{R}$ are members of the appropriate coverings of H_0 and $\mathbb{R}$ respectively. Then $t = (t_\theta, t_R) \in \ell \times \mathbb{R} = \sigma$, where $\ell \in \mathrm{H}_0$ is a boundary line, and $y = (y_\theta, y_R)$, where $y_\theta \in \widetilde{U}_\theta$, $y_R \in \widetilde{U}_\mathbb{R}$. For $t' = (t_\theta, y_R) \in \sigma$, we have

$$|xt'| \le |xt| + |tt'| \le |xt| + |ty| = |xy|.$$

If $t_\theta \in \widetilde{U}_\theta$, then $t' \in U \cap \sigma$ because $y_R \in \widetilde{U}_R$, and hence $|xy| \ge d(x, U \cap \sigma)$. Thus in what follows we assume that $t_\theta \notin \widetilde{U}_\theta$.

Let $\theta_p \times I_q$ be the initial brick in $\mathcal{W}_u^{(i,j)}$. Passing to the tree T_0, we denote by U_0 the set of all vertices q' in $\mathrm{V}(T_0)$ such that $\theta_{q'} \subset \widetilde{U}_\theta$. Then by the construction of $\mathcal{W}_u^{(i,j)}$, U_0 is the vertex set of a member of the standard (N,p)-covering of the tree T_0. Denote this member by U'. There is $k \in \mathbb{N} \cup \{0\}$ such that U' lies in the N-annulus

$$A_k = \{q \in T_0 \mid kN \le |qp| \le (k+1)N\}$$

centered at p. We denote by $\bar{\ell}$ the line in T_0 corresponding to ℓ and pick a vertex $p_0 \in \bar{\ell}$ of T_0 such that $t_\theta \in \theta_{p_0}$. There are at most two such vertices, and we assume that $p_0 \in U'$ if $t_\theta \in \widetilde{U}_\theta$. Then the assumption $t_\theta \notin \widetilde{U}_\theta$ is equivalent to $p_0 \notin U'$.

Since $y_\theta \in \widetilde{U}_\theta$, there is a vertex p_y of T_0 such that $y_\theta \in \theta_{p_y}$ and $p_y \in U'$. We show that $\mathrm{dist}(p_0, \bar{\ell} \cap U') \le |p_0 p_y|$ in T_0. The (unique) point $p' \in \bar{\ell}$ closest to p separates $\bar{\ell}$ into the rays $\bar{\ell}_i$, $i = 1,2$. Each of the rays $\bar{\ell}_1$, $\bar{\ell}_2$

intersects the annulus A_k over a segment, and at least one of them lies in U'. We assume without loss of generality that $\bar\ell_1 \cap A_k \subset U'$. There is a shortest segment $\bar\gamma \subset \bar\ell$ between p_0 and $\bar\ell \cap U'$. If $p_0 \in \bar\ell_1$, then $\bar\gamma \subset \bar\ell_1$ is a shortest path in T_0 between p_0 and U'. Thus the length $|\bar\gamma| = d(p_0, U')$ and hence $d(p_0, \bar\ell \cap U') \le |p_0 p_y|$.

It remains to consider the case $p_0 \in \bar\ell_2$ and $\bar\ell_2$ is disjoint with U' (the last happens exactly when $k > 1$ and $|pp'| \le (k - \frac{1}{2})N$). As above, we have $|p'p_y| \ge d(p', U') = d(p', \bar\ell \cap U')$. On the other hand, any path in T_0 between p_0 and U' passes over p'. Thus $|p_0 p_y| = |p_0 p'| + |p' p_y| \ge |\bar\gamma| = d(p_0, \bar\ell \cap U')$.

The geodesic segment $ty \subset X_u$ projects to the geodesic segment $t_\theta y_\theta$ in the factor H_0 of X_u. While moving from t_θ to y_θ along $t_\theta y_\theta$, one passes the segment $p_0 p_y \subset T_0$ and $|t_\theta y_\theta| \ge |p_0 p_y| - 1$ by $(\star)$.

Let $\gamma = t'z \subset \ell \times y_R \subset \sigma$ be the segment between t' and $z \in U \cap \sigma$ that corresponds to $\bar\gamma \subset \bar\ell$. The length of γ is at most the number of *interior* vertices of $\bar\gamma$ plus one, thus $|\gamma| \le |\bar\gamma| - 1$. Since $|p_0 p_y| \ge |\bar\gamma|$, we obtain $|t_\theta y_\theta| \ge |\gamma|$. It follows $|ty| \ge |tz|$ because both the triangles $tt'y$ and $tt'z$ are flat with right angles at t'. Then moreover $|xy| \ge |xz| \ge d(x, U \cap \sigma)$. □

Lemma 4. *Suppose σ is the common plane of the blocks X_u and $X_{u'}$ such that $|vu'| < |vu|$, where v is the root of T. Assume that $U \in \mathcal{W}^{(i,j)}$, $U \subset X_u$, is disjoint with σ, $U \cap \sigma = \emptyset$. Then $d(U, \sigma) \ge R$.*

Proof. Let $\theta_p \times I_q$ be the initial brick in $\mathcal{W}_u^{(i,j)}$. Recall that $U = \widetilde U_\theta \times \widetilde U_R$, where $\widetilde U_\theta$ and $\widetilde U_R$ are members of the appropriate coverings of the θ-tree H_0 and $\mathbb{R}$ respectively. Denote by U_0 the set of all vertices $q' \in \mathrm{V}(T_0)$ such that the hexagon $\theta_{q'} \in \widetilde U_\theta$. Then U_0 is the vertex set of a member U' of the standard (N, p)-covering of the tree T_0. There is $k \in \mathbb{N} \cup \{0\}$ such that U' is a subset of the N-annulus

$$A_k = \{q \in T_0 \mid kN \le |qp| \le (k+1)N\}$$

centered at p.

Representing $\sigma = \ell \times \mathbb{R}$, where $\ell \subset \mathrm{H}_0$ is an appropriate boundary line, we let $\bar\ell \subset T_0$ be the line corresponding to ℓ. It follows from the assumption on σ and the construction of the family $\mathcal{W}^{(i,j)}$ that $d(p, \bar\ell) \le N$ and thus $\bar\ell$ meets every annulus A_k, $k \ge 0$. By the assumption, ℓ is disjoint with $\widetilde U_\theta$, hence $\widetilde U_\theta$ is disjoint with any hexagon of H_0 that meets ℓ. Thus $\bar\ell$ is disjoint with U'. Since $\bar\ell \cap A_k \ne \emptyset$ is covered by members of the standard (N, p)-covering of T_0 having the same color $i \in \{1, 2\}$ as U' has and the covering is N-disjoint, we have $d(U', \bar\ell \cap A_k) \ge N$. Therefore $d(U', \bar\ell) \ge N/2$. Moreover, $N = 2R + 1$ is odd and the distance $d(U', \bar\ell)$ is integer, thus we actually have $d(U', \bar\ell) \ge (N+1)/2 = R + 1$. Using $(\star)$, we obtain $d(\widetilde U_\theta, \ell) \ge R$ and thus $d(U, \sigma) \ge R$. □

Proof of Proposition 1. Suppose that U lies in the block X_u and W lies in the block X_w. We prove the proposition by induction on $m = l(u) + l(w)$.

The base of induction : $m = 0$.
It follows from the assumption $d(U, W) < R$ that $U = W$, because different members of the family $\mathcal{W}^{(i,j)}$ are N-disjoint.

The induction step: $m \to m + 1$.
We assume that the assertion is true for all $l(u) + l(w) \le m$ and consider th

case $l(u) + l(w) = m + 1$. Without loss of generality suppose $l(u) \le l(w)$. Then $l(w) \ge 1$. Thus there exists a boundary plane σ in X_w such that $l(w') = l(w) - 1$ for the block $X_{w'}$, which is glued to X_w over σ.

We have $d(W, \sigma) < R$ because U, W are separated by σ. Therefore $W \cap \sigma \ne \emptyset$, since X_w and σ satisfy the conditions of Lemma 4. By Lemma 3, $d(x, W) = d(x, W \cap \sigma)$ for every $x \in U$, thus $d(U, W \cap \sigma) < R$. There is $W' \in \mathcal{W}^{(i,j)}$, $W' \subset X_{w'}$, such that $W' \cap \sigma = \sigma \cap W$ by construction of the family $\mathcal{W}^{(i,j)}$, in particular, W' is equivalent to W. Then $d(W', U) < R$, and thus W' is equivalent to U by the inductive assumption. Hence W is equivalent to U. $\square$

Given $(i, j) \in \{1, 2\}^2$, for each equivalence class $\widetilde{W}$ of the equivalence relation $\sim_{(i,j)}$ on the family $\mathcal{W}^{(i,j)}$ consider the set $W := \bigcup\{U \mid U \in \widetilde{W}\}$. We denote by $\widetilde{\mathcal{W}}^{(i,j)}$ the family of all such sets. Proposition 1 implies that the covering

$$\widetilde{\mathcal{W}} = \bigcup\{\widetilde{\mathcal{W}}^{(i,j)} \mid (i, j) \in \{1, 2\}^2\}$$

of X is 4-colored and R-disjoint. However, $\widetilde{\mathcal{W}}$ is unbounded, since the diameter of each of its members is infinite.

5. Proof of Theorem 1

5.1. **Extension of the covering members.** Let W be a member of the covering $\widetilde{\mathcal{W}}$. The sets $W_u = X_u \cap W$, $u \in \mathrm{V}(T)$, while nonempty, form the equivalence class $\widetilde{W}$ which corresponds to W. We have $W_u = U \times V$, where $U \subset \mathrm{H}_0$ and $V \subset \mathbb{R}$ are members of the coverings $\widetilde{\mathcal{U}}^i$ and $\widetilde{\mathcal{V}}^j$ respectively for some pair $(i, j) \in \{1, 2\} \times \{1, 2\}$.

Recall that U is the union of the hexagons labelled by all the vertices of the tree T_0 (see sect. 3.1 for definition of T_0) lying in a member U' of the standard (N, p)-covering of T_0. There is $k \in \mathbb{N} \cup \{0\}$ such that U' is contained in the N-annulus

$$A_k = \{q \mid kN \le |qp| \le (k + 1)N\} \subset T_0$$

centered at p.

Now, we construct an extension $\overline{U}'$ of U' as follows. For each $x \in U'$ consider the geodesic segment $\gamma_x = x'x \subset px$ of length $N/2$ (if $|px| < N/2$, then $k = 0$, $U' = B_N(p)$, and we define $\overline{U}' := U'$). We put

$$\overline{U}' := \bigcup\{\gamma_x \mid x \in U'\} \subset T_0$$

and define

$$\overline{U} := \bigcup\{\theta_v \mid v \in \overline{U}' \cap \mathrm{V}(T_0)\} \subset \mathrm{H}_0.$$

Note that the set $\overline{U}'$ is connected. Indeed, for each $x, y \in \overline{U}'$ there exist $x_1, y_1 \in U'$ such that $x \in \gamma_{x_1}, y \in \gamma_{y_1}$. Let s be the center of the tripod x_1, y_1, p. Definition of the set U' implies $|ps| \ge (k - \frac{1}{2})N$. There are $x' \in px_1$ and $y' \in py_1$ such that $|px'| = |py'| = kN$. Then x_1x', $y_1y' \subset U'$ and either $s \in U'$ or $|sx'|$, $|sy'| \le N/2$. In each case s is connected with x and y by paths in $\overline{U}'$, thus the set $\overline{U}'$ is connected. Then $\overline{U}$ is convex by Remark 3.

Recall that V is the union of the intervals labelled by all the vertices of the tree S_0 (homeomorphic to $\mathbb{R}$, see sect. 3.1 for definition of S_0) lying in

a member $V' \subset S_0$ of the standard (N,d,q)-covering of S_0 for some natural $d \le N$.

We similarly construct an extension $\overline{V}'$ of V'. For each $x \in V'$ consider the segment $\gamma_x := x'x \subset qx$ of length $N/2$ (if $|qx| < N/2$, then either $V' = B_d(q)$ and we define $\overline{V}' := V'$, or $V' = B_{d+N}(q) \setminus \operatorname{Int} B_d(q)$ and $d < N/2$, then we define $\overline{V}' := B_{d+N}(q)$). Consider the set

$$\overline{V}' := \bigcup \{\gamma_x \mid x \in V'\} \subset S_0$$

and define

$$\overline{V} := \bigcup \{I_v \mid v \in \overline{V}' \cap \mathrm{V}(S_0)\} \subset \mathbb{R}.$$

The set $\overline{V}'$ being a segment in S_0 is connected. Then $\overline{V}$ is convex by Remark 3.

Therefore $\overline{W}_u := \overline{U} \times \overline{V}$ is a convex subset of X_u. Moreover, $\overline{W}_u$ is compact by construction. We call it the *extension* of the set W_u. Note that similarly to Lemma 2 the extensions of the sets W_u and W_w agree for the blocks X_u and X_w having the common boundary plane σ, i.e. their intersections with σ coincide. We call the *extension* of the set W the union of the extensions of all its block components W_u.

Recall well known fact that for each point x in an Hadamard space X and for each closed convex subset $A \subset X$ there exists the unique point $p_x \in A$ such that $|xp_x| = d(x,A)$ (the point p_x is called the metric projection x to the set A). The map sending $x \in X$ to its metric projection $p_x \in A$ is called the *projection* to A and is a 1-Lipschitz map.

Proposition 2. *For each member W of the covering $\widetilde{\mathcal{W}}$ its extension $\overline{W}$ is convex.*

Proof. Given x, $y \in \overline{W}$, we prove that the geodesic segment $xy \subset \overline{W}$. Let X_u, $X_v \subset X$ $(u,v \in \mathrm{V}(T))$ be the blocks containing x, y respectively. If $u = v$, then x, $y \in \overline{W}_u$ that is convex. Thus we assume that $u \neq v$. The segment xy intersects consecutively the boundary planes $\sigma_1,\dots,\sigma_n$ of the blocks $X_0 = X_u,\dots,X_n = X_w$ $(\sigma_i = X_{i-1} \cap X_i$ for each $i \in \{1,\dots,n\})$ at $z_i \in \sigma_i$, $i \in \{1,\dots,n\}$. The sets $\overline{W}_i := X_i \cap \overline{W}$ are convex for all $i \in \{0,\dots,n\}$. Note that for each $i \in \{1,\dots,n\}$ the projection of z_i to $\overline{W}_{i-1}$ and to $\overline{W}_i$ coincide. Indeed, let $t \in \overline{W}_i$ be the nearest point to z_i. By construction of $\overline{W}_i$, the metric projection of $\overline{W}_i$ to σ_i coincides with $\overline{W}_i \cap \sigma_i$, in particular, the projection t_i of t to σ_i belongs to $\overline{W}_i$. Since σ_i is convex in the Hadamard space X, we have $|z_i t_i| \le |z_i t|$, therefore the nearest point in $\overline{W}_i$ to z_i lies in $\overline{W}_i \cap \sigma_i$. By similar argument with $\overline{W}_{i-1}$, we see that projections of z_i to $\overline{W}_{i-1}$ and to $\overline{W}_i$ coincide. Denote this point by z_i'. By convexity of $\overline{W}_i$, we have $|z_i' z_{i+1}'| \le |z_i z_{i+1}|$ for every $i = 0,\dots,n$, where $z_0 = z_0' = x$, $z_{n+1} = z_{n+1}' = y$, i.e. the curve $\gamma := xz_1' \cup \ldots \cup z_n' y$ has length less or equal to that of xy, hence $\gamma = xy$. Thus $xy \subset \overline{W}$. $\square$

5.2. **Additional colors.** Consider the extension $\overline{W}$ of a member W of the covering $\widetilde{\mathcal{W}}$. Note that for each block X_u such that $X_u \cap W \neq \emptyset$, the set $\overline{W}_u := X_u \cap \overline{W}$ is convex, compact and $17R$-bounded. The last follows from the fact that $W_u = X_u \cap W$ is $15R$-bounded and each point $x \in \overline{W}_u$ lies at the distance at most R from W_u.

Remark 4. Given $u, w \in V(T)$, any segment $xy \subset X$ with $x \in W_u$, $y \in W_w$ intersects $W_{u'}$ iff the segment $uw \subset T$ contains u'.

Put $c = 18$. Similarly to the standard coverings of a metric tree (see section 2), we construct for every member W of the covering $\widetilde{\mathcal{W}}$ a 2-colored, R-disjoint, $c'R$-bounded covering of the extension $\overline{W}$, where $c' := 5c - 2$.

Given $x \in X$, $u \in V(T)$, we use notation $x\overline{W}_u$ for the geodesic segment $xx' \subset X$, where x' is the metric projection of x to $\overline{W}_u$. This is well defined because $\overline{W}_u$ is convex and thus x' is unique. We fix $v \in \mathrm{V}(T)$ such that $W \cap X_v \neq \emptyset$. For every $k \in \mathbb{N} \cup \{0\}$ consider the "annulus"

$$A_k := \bigcup \{\overline{W}_u \mid ckR \leq d(\overline{W}_v, \overline{W}_u) \leq c(k+1)R\} \subset \overline{W}.$$

We define a relation $\sim_k$ on the set A_k saying that $x \sim_0 y$ for every x, $y \in A_0$, and for each natural k we put $x \sim_k y$ iff there exists a vertex $u \in \mathrm{V}(T)$ such that the geodesic segments $x\overline{W}_v$, $y\overline{W}_v$ intersect the set $\overline{W}_u$ and

$$d(\overline{W}_u, \overline{W}_v) \geq (c(k-1) + 1/2)R.$$

Note that $x \sim_k y$ for each x, $y \in \overline{W}_u \subset A_k$.

We prove that $\sim_k$ is an equivalence relation. Indeed, suppose $x \sim_k y$ and $y \sim_k z$. Given $x \in \overline{W}_{v_x}, y \in \overline{W}_{v_y}$, $z \in \overline{W}_{v_z}$, there are $u \in \mathrm{V}(T)$ from the definition of $\sim_k$ for the pair $\{x, y\}$ and $w \in \mathrm{V}(T)$ for the pair $\{y, z\}$. Note that u, w lie on the segment $v_y v \subset T$. Without loss of generality suppose that $|uv| \leq |wv|$. Then the segment $z\overline{W}_v$ intersects the set $\overline{W}_u$, so the points x and z are equivalent too.

Consider the following 2-colored covering of the set $\overline{W}$: the sets of the first color are all the equivalence classes of $\sim_k$, where k is even, the sets of the second color are all the equivalence classes of $\sim_k$, where k is odd.

We prove that this covering of $\overline{W}$ is $c'R$-bounded. Indeed, assume that $x \in \overline{W}_{v_x} \subset A_k$, $y \in \overline{W}_{v_y} \subset A_k$ are equivalent, and let $u \in \mathrm{V}(T)$ be a vertex such that the segments $x\overline{W}_v$, $y\overline{W}_v$ intersect the set $\overline{W}_u$ and

$$d(\overline{W}_u, \overline{W}_v) \geq (c(k-1) + 1/2)R.$$

For z, $t \in \overline{W}_u$ with $z \in x\overline{W}_v$, $t \in y\overline{W}_v$ we have $|z\overline{W}_v|$, $|t\overline{W}_v| \geq (c(k-1) + 1/2)R$. Then $|xz| = |x\overline{W}_v| - |z\overline{W}_v| \leq (2c - \frac{1}{2})R$, and similarly $|yt| \leq (2c - \frac{1}{2})R$. Since $\overline{W}_u$ is $(c-1)R$-bounded, we obtain

$$|xy| \leq |xz| + |zt| + |ty| \leq (5c - 2)R = c'R.$$

Now, we prove that the constructed covering of $\overline{W}$ is R-disjoint. Let $U_1 \subset A_k, U_2 \subset A_l$ be sets of the same color.

Assume first that $|k - l| \geq 2$. Without loss of generality suppose $k \geq l + 2$. If $d(U_1, U_2) < R$, then $|xy| < R$ for some $x \in \overline{W}_{v_x} \subset U_1$, $y \in \overline{W}_{v_y} \subset U_2$. Note that $d(\overline{W}_v, \overline{W}_{v_y}) \leq c(l+1)R$, thus $|zt| \leq c(l+1)R$ for some $z \in \overline{W}_{v_y}$, $t \in \overline{W}_v$. Then

$$|xt| \leq |xy| + |yz| + |zt| < R + (c-1)R + c(l+1)R = c(l+2)R \leq ckR.$$

This contradicts $ckR \leq d(\overline{W}_v, \overline{W}_{v_x})$.

Assume now that $k = l$, $\overline{W}_a \subset U_1$, $\overline{W}_b \subset U_2$. It suffices to prove that $d(\overline{W}_a, \overline{W}_b) \geq R$. Let $u \in \mathrm{V}(T)$ be the center of the tripod $abv \subset T$. Then

any geodesic segment from $\overline{W}_a$, $\overline{W}_b$ to $\overline{W}_v$ passes through $\overline{W}_u$ and

$$d(\overline{W}_u, \overline{W}_v) < (c(k-1)+1/2)R.$$

We show that $d(\overline{W}_a, \overline{W}_u) \geq R/2$. Indeed, suppose that it is not true. Then there are $x_1 \in \overline{W}_a$, x_2, $z_2 \in \overline{W}_u$, $z_1 \in \overline{W}_v$ such that $|x_1x_2| < R/2$, $|z_1z_2| < (c(k-1)+1/2)R$. Thus

$$|x_1z_1| \leq |x_1x_2| + |x_2z_2| + |z_2z_1| < R/2 + (c-1)R + (c(k-1)+1/2)R = ckR$$

in contradiction with $\overline{W}_a \subset A_k$.

Similarly, we have $d(\overline{W}_b, \overline{W}_u) \geq R/2$. It follows from $u \in ab \subset T$ that

$$d(\overline{W}_a, \overline{W}_b) \geq d(\overline{W}_a, \overline{W}_u) + d(\overline{W}_u, \overline{W}_b) \geq R.$$

We restrict the constructed covering of $\overline{W}$ to $W \subset \overline{W}$, i.e. for each member of the covering consider its intersection with W. The new covering of W is again 2-colored, R-disjoint and $c'R$-bounded with $c' = 5c - 2 = 88$. We denote this covering by $\mathcal{X}_W$, $\mathcal{X}_W = \mathcal{X}_W^1 \cup \mathcal{X}_W^2$.

For each triple of numbers $(i,j,k) \in \{1,2\}^3$ consider the following 8-colored covering of X,

$$\mathcal{X} := \bigcup \{\mathcal{X}_W^k \mid W \in \widetilde{\mathcal{W}}^{(i,j)}, \quad (i,j,k) \in \{1,2\}^3\}.$$

The covering $\mathcal{X}$ is $c'R$-bounded because the families $\mathcal{X}_W^k$, $k = 1,2$, are $c'R$-bounded. We prove that $\mathcal{X}$ is R-disjoint. Indeed, let $U \in \mathcal{X}_{W_1}^k, V \in \mathcal{X}_{W_2}^k$ be two different sets of the same color for some triple of numbers $(i,j,k) \in \{1,2\}^3$, where $W_1, W_2 \in \widetilde{\mathcal{W}}^{(i,j)}$. If $W_1 \neq W_2$, then $d(U,V) \geq d(W_1,W_2) \geq R$, since $U \subset W_1$, $V \subset W_2$, and the families $\widetilde{\mathcal{W}}^{(i,j)}$ are R-disjoint. If $W_1 = W_2$, then $d(U,V) \geq R$ because $\mathcal{X}_{W_1}^k$ is R-disjoint. This completes the proof of Theorem 1.

6. Applications

Let us recall some definitions.

Definition 3. (Assouad [1]) *The Assouad – Nagata dimension* of a metric space X, $\dim_{\mathrm{AN}} X$, is the least integer number n such that for each $R > 0$ there exists an $(n+1)$-colored, R-disjoint, CR-bounded covering of the space X, where the number $C > 0$ is independent from R.

Definition 4. (Buyalo [6]) *The linearly-controlled dimension* of a metric space X, $\ell\text{-}\dim X$, is the least integer number n, such that for each $R \leq 1$ there exists an $(n+1)$-colored, R-disjoint, CR-bounded covering of the space X, where the number $C > 0$ is independent from R.

One easily sees that $\dim_{\mathrm{AN}} X = \max\{\ell\text{-}\dim X, \ell\text{-asdim}\, X\}$.

Recall that $\widetilde{M}$ is the universal cover of the graph-manifold M. A remark to [9, Proposition 2.7] implies that $\ell\text{-}\dim \widetilde{M} = \dim \widetilde{M}$, so $3 = \dim \widetilde{M} \leq \dim_{\mathrm{AN}} \widetilde{M}$. Theorem 1 implies that $\dim_{\mathrm{AN}} \widetilde{M} \leq 7$, which gives Corollary 1. However, by Bell-Dranishnikov result (see [3]) $\operatorname{asdim} \widetilde{M} = 3$, which suggests following conjecture.

Hypothesis. *Let $\widetilde{M}$ be the universal cover of the graph-manifold. Then* $\dim_{\mathrm{AN}} \widetilde{M} = 3$.

Corollary 2 follows from Corollary 1 and [9, Theorem 1.3].

Definition 5. A metric space X called *Lipschitz n-connected*, $n \geq 0$, if for each $m \in \{0, \dots, n\}$ there exists $\gamma > 0$ such that every λ-Lipschitz map $f\colon S^m \to X$ permits a $\gamma\lambda$-Lipschitz extension $\bar{f}\colon B^{m+1} \to X$, where S^m and B^{m+1} are the unit sphere and the unit disk in $\mathbb{R}^{m+1}$ respectively.

It is proved in [4], that Hadamard spaces are Lipschitz n-connected for each $n \geq 0$. Thus the model space X is Lipschitz n-connected for each $n \geq 0$. By Corollary 1 the space X has finite Assouad-Nagata dimension. Corollary 3 now follows from [9, Theorem 1.5].

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